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COMMON FIXED POINT THEOREMS IN CONE METRIC SPACES

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Abstract

In this paper we established a fixed point theorem for multivalued contractive mappings in cone metric space which generalize the common unique fixed point theorem to the case of multivalued mappings in cone metric space. Our results are the extensions of the results obtained by Mohammad *et.al* [10] to the case of cone metric spaces.

Keywords: fixed point,cone metric spaces, Common fixed point, contractive multivalued mappings.

1. Introduction:

Cone metric spaces were introduced in [6]. After carefully defining convergence and completeness in cone metric spaces, the authors proved some fixed point theorems of contractive mappings. Recently,many authors have established and extended different type of contractive mappings in cone metric spaces see for instances [3], [4], [7], [9], The author [3] have also proved fixed point in cone metric space for generalized contractive mappings. The purpose of this paper is extension and proves the common unique fixed point of the results [3],[6] and [10].Our results extend the various comparable results in literature [4], [6], and [7] First, we recall some standard notations and definitions in cone metric spaces with some of their properties [3].

2.Preliminaries.

We recall the definitions and example as well as some properties of cone metric spaces which are necessary for a good understanding of the work below.

Definition 2.1.[6].Let E be a real Banach space and P a subset of E. P is called a cone if and only if

(i) P is closed, nonempty, and $P \neq \{0\}$,

(ii) $ax + by \in P \; \forall \; x, y \in P$ and non-negative real number a, b;

(iii) $x \in P$and $-x \in P \Longrightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ if and only if $y - x \in Int\, P$, Int P denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq K \leq \|y\|$. The least positive number satisfying above is called the normal constant of P.

In the following, we always suppose E is a Banach space, P is a cone in E with Int $P \neq \varphi$ and $\leq$ is partial ordering with respect to P.

Definition 2.2. [6] - Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies

(i) $0 \leq d(x, y) \; \forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x) \forall x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$

Then d is called cone metric on X, and (X, d) is called a cone metric space.

Example 2.3. Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\} \subset R^2, X = R$ and

$d : X \times X \to E$such that $d(x, y) = (| x - y |, \infty| x - y |)$, where $\infty > 0$ is a constant. Then (X, d) is cone metric space.

Definition 2.4. [6]. Let (X, d)be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with$0 \ll c$, there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x \; or \; x_n \to x \; as \; n \to \infty$.

Definition 2.5. [6] - Let (X, d)be a cone metric space, $\{x_n\}$be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N, d\,(x_n, x_m)$ ≪c, then $\{x_n\}$ is called a Cauchy sequence in X.

Lemma 2.6. [6]- Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$converges to x if and only if $d(x_n, x) \to o \; as \; n \to \infty$).

(ii) is a Cauchy sequence if and only if $d(x_n, y_n) \to o \; as \; n, m \to \infty$).

Proof. For every $\in > 0$, choose $c \in E$ with $0 \ll c$ and $\|c\| < \frac{\in}{4k+2}$. From $x_n \to x$ and $y_n \to y$, there is N such that for all $n > N$,

$d(x_n, x) \ll c$ and $d(y_n, y) \ll c$.

We have

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y_n, y) \le d(x, y) + 2c,$$

$$d(x, y) \le d(x_n, x) + d(x_n, y_n) + d(y_n, y) \le d(x_n, y_n) + 2c.$$

Hence

$$0 \le d(x, y) + 2c - d(x_n, y_n) \le 4c.$$

and

$$\|d(x_n, y_n) - d(x, y)\| \le d(x, y) + 2c - d(x_n, y_n) + \|2c\| \le (4k + 2)\|c\| < \in.$$

Therefore $d(x_n, y_n) \to d(x, y) as \; n \to \infty$.

Definition 2.7. [3]Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 2.8.[6]Let (X, d)be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$and $\{y_n\}$be two sequences in X;

(i) If $\{x_n\}$converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique, obviously limit of $\{y_n\}$is also unique.

(ii) If $x_n \to x, y_n \to y \; as \; n \to \infty. \; Then \; d(x_n, y_n) \to d(x, y) \; as \; n \to \infty)$.

2. MAIN RESULTS

Theorem 3.1. Let(X, d) be a complete cone metric space. Suppose that $T_1, T_2: (X, d) \to (X, d)$ be any two contractive multivalued mapping satisfying:

$$F[H(T_1x, T_2y) \leq \alpha[F\{d(x, y) + d(T_1x, T_2y)\}] + \beta[F\{d(x, T_1x) + d(y, T_2y)\}]$$

$$+\gamma[F\{(x, T_2y) + d(y, T_1x)\}] + \delta \frac{F\{d(x, T_2y) + d(T_1x, T_2y)\}}{1 - F\{d(x, T_2y)d(T_1x, T_2y)\}}$$

$\forall \; x, y \in X, \alpha + \beta + \gamma + \frac{1}{2}\delta < \frac{1}{2}; \; \alpha, \beta, \gamma \in \left[0, \frac{1}{2}\right]$ and $1 - F\{d(x_n, T_1x^*)d(T_1x_n, T_1x^*)\} > 0$.

Then T_1andT_2have a unique common fixed point in X. For each $x, y \in X$, the iterative sequence $\{{T_1}^n x\}$converges to the unique fixed point.

Proof: For all $x_0 \in X$ and $n \geq 1, x_1 \in T_1(x_0), \ldots \ldots \ldots \ldots \ldots \ldots, x_{n+1} \in T_1(x_n)$by iteration method we have a sequence $\{x_n\}$ of unique point in X by letting

$$x_1 = T_1(x_0), x_2 = T_1(x_2) = {T_1}^2 x_0, \ldots \ldots \ldots \ldots \ldots \ldots., x_{n+1} = T_1 x_n = {T_1}^{n+1} x_0, \ldots \ldots \ldots \ldots \ldots \ldots..$$

Then

$$F[d(x_{n+1}, x_n) \leq F[H(T_1x_n, T_2x_{n-1})$$

$$\leq \alpha[F\{d(x_n, x_{n-1}) + d(T_1x_n, T_2x_{n-1})\}] + \beta[F\{d(x_n, T_1x_n) + d(x_{n-1}, T_2x_{n-1})\}]$$

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$$+\gamma[F\{d(x_n,T_2x_{n-1})+d(x_{n-1},T_1x_n)\}]+\delta\frac{F\{d(x_n,T_2x_{n-1})+d(T_1x_n,T_2x_{n-1})\}}{1-F\{d(x_n,T_2x_{n-1})d(T_1x_n,T_2x_{n-1})\}}$$

$$\leq \alpha[F\{d(x_n,x_{n-1})+d(x_{n+1},x_n)\}]+\beta[F\{d(x_n,x_{n+1})+d(x_{n-1},x_n)\}]$$

$$+\gamma[F\{d(x_n,x_n)+d(x_{n-1},x_{n+1})\}]+\delta\frac{F\{d(x_n,x_n)+d(x_{n+1},x_n)\}}{1-F\{d(x_n,x_n)d(x_{n+1},x_n)\}}$$

$$\leq \alpha[F\{d(x_n,x_{n-1})+d(x_{n+1},x_n)\}]+\beta[F\{d(x_n,x_{n+1})+d(x_{n-1},x_n)\}]$$

$$+\gamma[F\{d(x_{n-1},x_{n+1})\}]+\delta[F\{d(x_{n+1},x_n)\}]$$

$$\leq \alpha[F\{d(x_n,x_{n-1})+d(x_{n+1},x_n)\}]+\beta[F\{d(x_n,x_{n+1})+d(x_{n-1},x_n)\}]$$

$$+\gamma[F\{d(x_n,x_{n+1})+d(x_{n-1},x_n)\}]+\delta[F\{d(x_{n+1},x_n)\}]$$

$$\leq (\alpha+\beta+\gamma)F[d(x_n,x_{n+1})+d(x_n,x_{n-1})]+\delta[F\{d(x_{n+1},x_n)\}]$$

$\Rightarrow F[d(x_{n+1},x_n)\leq\left(\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta+\gamma+\delta}\right)F[d(x_n,x_{n-1})]$ where $\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta+\gamma+\delta}=S$

Hence $F[d(x_{n+1},x_n)=S^nF[d(x_1,x_0)]$ for $n>m$ we have

$$F[d(x_n,x_m)\leq F[d(x_n,x_{n-1})+d(x_{n-1},x_{n-2})]+\cdots\dots\dots\dots\dots\dots\dots.\dots\dots..+d(x_{m+1},x_m)$$

$$\leq[S^{n-1}+S^{n-2}+S^{n-3}+S^{n-4}+\ \dots\dots\dots\dots\dots\dots\ +S^m]F\{d(x_1,x_0)\}$$

$$\leq\frac{S^m}{1-S}F\{d(x_1,x_0)\}$$

For a natural number N_1. Let $d<0$ such that $\frac{S^m}{1-S}F\{d(x_1,x_0)\}<d,\forall m\geq N_1$. Thus $d(x_n,x_m)<d$ for $n>m$.

Therefore$\{x_n\}$ is a Cauchy sequence in X. Since (X,d) be complete cone metric space, $\exists\ x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. Choose a natural number N_2 such that$F\{d(x_{n+1},x_n)\}<(1-t)\frac{d}{3}$ and $F\{d(x_{n+1},x^*)\}<(1-t)\frac{d}{3},\forall\ n\geq N_2$. We have

$$F[d(T_1x^*,x^*)\leq F[H(T_1x_n,T_1x^*)+d(T_1x_n,x^*)]$$

$$\leq \alpha[F\{d(x_n, x^*) + d(T_1x_n, T_1x^*)\}] + \beta[F\{d(x_n, T_1x_n) + d(x^*, T_1x^*)\}]$$

$$+\gamma[F\{d(x_n, T_1x^*) + d(T_1x_n, x^*)\}] + \delta \frac{F\{d(x_n, T_1x^*) + d(T_1x_n, T_1x^*)\}}{1 - F\{d(x_n, T_1x^*)d(T_1x_n, T_1x^*)\}} + F[d(T_1x_n, x^*)]$$

$$\leq \alpha[F\{d(x_n, x^*) + d(x_{n+1}, T_1x^*)\}] + \beta[F\{d(x_n, x_{n+1}) + d(x^*, T_1x^*)\}]$$

$$+\gamma[F\{d(x_n, T_1x^*) + d(x_{n+1}, x^*)\}] + \delta \frac{F\{d(x_n, T_1x^*) + d(x_{n+1}, T_1x^*)\}}{1 - F\{d(x_n, T_1x^*)d(x_{n+1}, T_1x^*)\}} + F[d(x_{n+1}, x^*)]$$

$$\leq \alpha[F\{d(x_n, x^*) + d(x_{n+1}, T_1x^*)\}] + \beta[F\{d(x_n, x_{n+1}) + d(x^*, T_1x^*)\}] + \gamma[F\{d(x_n, T_1x^*) + d(x_{n+1}, x^*)\}] + F[d(x_{n+1}, x^*)]$$

$$\leq \alpha[F\{d(x_n, x^*) + d(x_{n+1}, x^*) + d(x^*, T_1x^*)\}] + \beta[F\{d(x_n, x^*) + d(x^*, x_{n+1}) + d(x^*, T_1x^*)\}] + \gamma[F\{d(x_n, x^*) + d(x^*, T_1x^*) + d(x_{n+1}, x^*)\}] + F[d(x_{n+1}, x^*)]$$

$$\Rightarrow (1-t)F[d(T_1x^*, x^*)] \leq t[F\{d(x_n, x^*)\}] + t[F\{d(x_{n+1}, x^*)\}] + F[d(x_{n+1}, x^*)]$$

$$\leq [F\{d(x_n, x^*)\}] + [F\{d(x_{n+1}, x^*)\}] + F[d(x_{n+1}, x^*)]$$

Where, $t = \alpha + \beta + \gamma$

$$\Rightarrow F[d(T_1x^*, x^*)] \leq \frac{[F\{d(x_n, x^*)\}] + [F\{d(x_{n+1}, x^*)\}] + F[d(x_{n+1}, x^*)}{(1-t)}.$$

$$\leq \frac{d}{3} + \frac{d}{3} + \frac{d}{3} = d, n \geq N_1$$

Thus $F[d(T_1x^*, x^*)] \leq \frac{d}{m}, \forall m \geq 1$, so $\frac{d}{m} - F[d(T_1x^*, x^*)] \in P, \forall m \geq 1$.Since $\frac{d}{m} \to 0$ as $m \to \infty$ and P is $-F[d(T_1x^*, x^*)] \in P$. But $d(T_1x^*, x^*) \in P$. Therefore, $d(T_1x^*, x^*) \in P = 0$ and so, $T_1x^* = x^*$. Now if $x^{**}$is another fixed point of T_1.

Then $F[d(x^*, x^{**})] = F[d(T_1x^*, T_1x^{**})] \leq t[d(T_1x^*, x^*) + d(T_1x^{**}, x^{**}] = 0$.

Hence $x^* \in T_1x^* = x^* \in T_1x^{**}$. Therefore, x^* is a unique fixed point of T_1.

Similarly, it can be established that $x^* \in T_1 x^* = x^* \in T_2 x^*$. Thus $x^*$is the common unique fixed point of T_1 and T_2.

Theorem:3.2. Let (X, d) be a cone metric space and let $T_1, T_2: (X, d) \to CB(X)$ be any two multivalued mappings satisfying $x, y \in X$,

$$F[H(T_1 x, T_2 y)] \le q\ max\ [F\{d(x, y), d(x, T_1 x), d(y, T_2 y), d(T_1 x, T_2 y)\}]$$

$\forall x, y \in X, and\ q \in [0, 1]$.

Then $T_1\ and\ T_2$ has a common fixed point in X. For each $x, y \in X$, the iterative sequence $\{{T_1}^n x\}$converges to the fixed point.

Proof: For each $x_0 \in X$and $n \ge 1$, $x_0 \in T_1(x_0), \dots\dots\dots, x_{n+1} \in T_1(x_n)$ by iteration method, we have a sequence $\{x_n\}$of point in Xby letting $x_1 = T_1 x_0, x_2 = T_1 x_1 = {T_1}^2 x_0, \dots\dots\dots\dots, x_{n+1} = T_1 x_n = {T_1}^{n+1} x_0, \dots\dots\dots$

Then,

$$F[d(x_{n+1}, x_n)] \le F[H(T_1 x_n, T_2 x_{n-1})]$$

$$\le q\ \max[F\{d(x_n, x_{n-1}), d(x_n, T_1 x_n), d(x_{n-1}, T_2 x_{n-1}), d(T_1 x_n, T_2 x_{n-1})\}]$$

$$\le q\ max[F\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_n)\}]$$

$$\le q\ max[F\{d(x_{n-1}, x_n), d(x_{n+1}, x_n)\}]$$

$$\le q[F\{d(x_n, x_{n-1})\}]$$

$$\Longrightarrow F[d(x_{n+1}, x_n)] \le q^n F[d(x_1, x_0)]$$

For $n > m$ we have

$$F[d(x_n, x_m)] \le F[d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots\dots\dots\dots\dots + d(x_{m+1}, x_m)$$

$$\le F[q^{n-1} + q^{n-2} + q^{n-3} + q^{n-4} + \dots\dots\dots\dots + q^m]F[d(x_1, x_0)]$$

$$\le \frac{q^m}{1-q} F[d(x_1, x_0)]$$

For a natural number N_1 let $d < 0$ such that$\frac{q^m}{1-q} d(x_1, x_0) < d, \forall\ m \ge N_1$.

Thus $d(x_n, x_m) < d$for $n > m$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\exists\ x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Choose a natural number N_2 such that $d(x_{n+1}, x_n) < (1-t)\frac{d}{3}$ and $d(x_{n+1}, x^*) < (1-t)\frac{d}{3}, \forall\ n \ge N_2$. We have

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$$F[d(T_1x^*, x^*)] \leq F[H(T_1x_n, T_1x^*) + d(T_1x_n, x^*)$$
$$\leq q\, max[\mathrm{F}\{d(x_n, x^*), d(x_n, T_1x_n), d(x^*, T_1x^*), d(T_1x_n, T_1x^*)\}] + F[d(T_1x_n, x^*)]$$
$$\leq q\, max\, [F\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, T_1x^*), d(x_{n+1}, T_1x^*)\}] + F[d(x_{n+1}, x^*)]$$
$$\leq q\, max\, [F\{d(x_n, x^*), d(x_n, x^*)\} + F\{d(x^*, x_n)d(x^*, T_1x^*)\}]$$

$\Rightarrow F[d(T_1x^*, x^*)] \leq d, \forall\, n \geq N_1]$. Thus $F[d(T_1x^*, x^*)] \leq \frac{d}{m}, \forall\, m \geq 1$, So,$\frac{d}{m} - F[d(T_1x^*, x^*)] \in P, \forall m \geq 1$. Since $\frac{d}{m} \to 0$ as $m \to \infty$and P is closed, $-d(T_1x^*, x^*) \in P$. But, $d(T_1x^*, x^*) \in P$. Therefore, $d(T_1x^*, x^*) \in P = 0$and so, $T_1x^* = x^*$. Now if x^{**} is another fixed point of T_1. Then $F[d(x^*, x^{**})] = F[d(T_1x^*, T_1x^{**})] \leq t[d(T_1x^*, x^*) +$ *dT1x∗∗, x∗=0.*

Hence $x^* \in T_1x^* = x^* \in T_1x^{**}$. Therefore, x^* is a fixed point of T_1.

Similarly, it can be established that $x^* \in T_1x^* = x^* \in T_2x^*$. Thus x^* is the common fixed point of $T_1$and T_2.

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